MS&E 319: Matching Theory
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HW#2 – Due Friday May 20, 2022

Please complete two of the following three problems for full credit. You can complete all three problems for extra credit.

1. Consider the following generalization of online bipartite matching to a weighted setting. We are given a set of offline nodes upfront; online nodes arrive one-by-one, revealing their *weights* to all offline nodes, and must be matched irrevocably upon arrival.

   (a) Show that (in contrast to the unweighted setting) it is impossible to design an algorithm that achieves a constant factor approximation to the weight of the optimum matching in hindsight.

   (b) With this difficulty in mind, we introduce the *free disposal* assumption: each offline node i can be matched as many times as we like, but only the highest weight edge counts to the objective (i is "disposing" of its lower-weight edges). Show that with the free disposal assumption, an online algorithm can achieve a 1/2-approximation to the optimum matching in hindsight.

2. Consider the following ubiquitous problem: whenever presented with one of an unknown number of tasks, you can either rent equipment needed for the task, at a cost, or buy the equipment for B times the cost of renting, following which you no longer need to rent for future tasks. Had you known the number of tasks upfront, minimizing the overall cost would have been trivial. Here we explore how to keep the cost low even in an online setting.

   (a) Formulate an LP capturing this problem (after relaxing integrality constraints). Take this LP's dual.

   (b) Provide a fractional online rent/buy algorithm with competitive ratio tending to $e/(e-1)$ as $B \to \infty$.

   (c) Provide a randomized (integral) rent/buy algorithm with competitive ratio tending to $e/(e-1)$ as $B \to \infty$.

3. Consider the following online matching problem: you are given a bipartite graph $G = (T \sqcup I, E)$ for $T = \{1, 2, \dots, n\}$ and a fractional matching $\{x_e\}_{e \in I}$. Each edge $e \in E$ is active with probability x_e and the edges incident to a single $t \in T$ are *correlated* so that at most one is active. In particular, for every vertex $t \in T$, a random subset of edges $A_t \subseteq \{it\}_{i \in I}$ is active such that $|A_t| \leq 1$ with probability 1 and $\Pr[it \in A_t] = x_{it}$ for all i. The random subsets $\{A_t\}_{t \in T}$ are independently realized for different $t \in T$.

   At time $t = 1, \dots, n$, online node t arrives and reveals which incident edges are active. You must choose irrevocably whether to match along the active edge if it exists (possibly in a randomized way).

   Design an online algorithm such that every edge e is matched with probability at least $x_e/2$. Show also there is no constant $c > 1/2$ such that we can match every edge e with probability $c \cdot x_e$.